A Quantitative Analysis of Optimal Sustainable Monetary Policies

Takeki Sunakawa*

Abstract

This study examines the quantitative properties of optimal sustainable monetary policies using a monetary model with a stabilization bias. As in Kurozumi (2008), the optimal sustainable policy is a strategy considered in the absence of commitment technologies; however it is implemented following an optimal quasi-sustainable policy derived by assuming that the commitment technologies are present. This study finds that solving for the policy function of the optimal quasi-sustainable policy yields a result basically identical to the Ramsey-optimal commitment policy under a set of parameters commonly used in the literature. The simulation shows two further results: policymakers have incentive to deviate from the Ramsey-optimal commitment policy when the lagged output gap is large and the optimal quasi-sustainable policy endogenously diminishes the steadfastness of policymakers' commitment.

Keywords: Optimal monetary policy; Time inconsistency; Sustainable plans; Timeless perspective

*Graduate School of Public Policy, the University of Tokyo. E-mail address: sunakawa@pp.u-tokyo.ac.jp.

1 Introduction

In New Keynesian (NK) models used in recent monetary policy analysis, the Ramsey-optimal commitment policy usually yields higher social welfare than an optimal discretionary policy, in which policymakers cannot commit to future policies.[1] Time-inconsistency plagues the optimal commitment policy (Kydland and Prescott, 1977; Barro and Gordon, 1983) because policymakers are tempted to abandon their previously announced policy and exploit the private sector's expectation.

To overcome the time-inconsistency problem, Kurozumi (2008) proposes the optimal sustainable monetary policy by analyzing Chari and Kehoe's (1990) sustainable plans in an NK model. The optimal sustainable monetary policy is based on the more realistic assumption that policymakers cannot access commitment technologies but can use reputation among the private sector. Therefore, when considering conduct of monetary policy, the optimal sustainable monetary policy is an important alternative to the Ramsey-optimal commitment policy. However, Kurozumi's analysis considers only qualitative properties; i.e., it merely checks whether policymakers have incentives to deviate from the optimal commitment policy.

This study presents a quantitative analysis of optimal sustainable monetary policies. As stated in Kurozumi (2008), *the optimal sustainable policy* is a strategy for the best sustainable equilibrium in the *absence* of commitment technologies.[2] However, it is implemented by following *the optimal quasi-sustainable policy* derived from the Lagrange method of Marcet and Marimon (1994, revised in 1998 and 2011) assuming the *presence* of commitment technologies. This study applies a version of the policy function iteration method in Kehoe and Perri (2002) to solve for the policy function of the optimal quasi-sustainable policy. While doing this, it examines the equilibrium dynamics of optimal sustainable monetary policies, such as impulse response functions and stochastic simulations.

This study finds that the optimal commitment policy is sustainable (i.e., the optimal quasi-sustainable policy yields a result basically identical to the optimal commitment policy) given a particular parameter set and a price markup shock as calibrated from U.S. data and commonly found in the literature.[3] This finding contrasts to Kurozumi's finding that the optimal commitment policy is not sustainable for some plausible parameters.

[1] We assume that the optimal commitment policy considered here is under *time-0* perspective. See a discussion later in the introduction.

[2] The optimal sustainable policy exploits an explicit punishment for the deviation from the current policy. This out-of-equilibrium punishment in the optimal sustainable policy serves as the commitment technologies in the optimal quasi-sustainable policy.

[3] See Rotemberg and Woodford (1998), Woodford (2003), Adam and Billi (2006, 2007), Nakov (2008), Giannoni (2010), and Bodenstein, Hebden, and Nunes (2012).

There are two types of the optimal commitment policy in the literature. No commitment is made in the past at an initial period t_0. If the policymaker made commitment in period $t_0 > -\infty$, the optimal commitment policy is under *time-0* perspective. In contrast, such an initial period is set in the infinite past under *timeless* perspective (Woodford, 1999). Dennis (2010) and Sauer (2010a,b) show that the optimal discretionary policy can be superior to the optimal commitment policy under the timeless perspective when the lagged output gap is large. The lagged output gap measures policymakers' previous commitment; the larger the gap, the greater are policymakers' incentives to deviate from that commitment. In this study, the optimal quasi-sustainable policy endogenously diminishes the steadfastness of their commitment when the lagged output gap is large. In an economy with some alternative parameters, the optimal quasi-sustainable policy yields higher social welfare than the optimal commitment policy in terms of the unconditional expected utility. This occurs because the optimal quasi-sustainable policy diminishes the steadfastness of policymakers' commitment and reduces the volatility of inflation and the output gap.

Related literature

Drawing upon game theory, Chari and Kehoe (1990) propose the concept of sustainable plans that become optimal in dynamic decision-making for a player who cannot commit to future plans in the present. Their standard is subgame perfect equilibria in repeated games between a strategic player and an infinite number of small agents. Chari and Kehoe (1990) follow Abreu (1988) in using the worst sustainable equilibrium to characterize the entire set of sustainable equilibria. The best sustainable equilibrium is then supported by the reputation of the strategic player among small agents. The equilibrium condition is summarized by *the sustainability constraint,* an inequality implying that any sustainable equilibrium outcome is above the worst.

The source of the time inconsistency in NK models is not only the well-known inflation bias but also a stabilization bias following price markup shocks and a trade-off between stabilizing inflation and stabilizing the output gap. Kurozumi (2008) proposes the notion of optimal sustainable monetary policy after analyzing Chari and Kehoe's sustainable plans in an NK model. He examines stabilization bias with an infinite length of punishment under the optimal discretionary policy and finds that the sustainability constraint is binding for some plausible parameters and the upper bound of the markup shock. Loisel (2008) considers inflation bias and stabilization bias with a finite duration of punishment under the optimal discretionary policy and concludes that both can be overcome (i.e., the sustainability constraint is not binding) by the reputation of the policymaker when the duration is a few years. However, Kurozumi (2008) and Loisel (2008) examine only whether the sustainability constraint is binding and disregard the optimal sustainable policy itself.

Woodford (1999) overcomes time-inconsistency in the Ramsey-optimal commitment policy by disregarding time—i.e., by assuming policymakers committed to future policies in the indeterminate past. The optimality condition in the initial period then can be ignored and a time-invariant policy rule can be derived. However, when the policies are evaluated according to policymaker's objectives from date $t_0 > -\infty$ forth, and taking the state of the economy at date t_0 as given, the optimal discretionary policy can be superior to the optimal commitment policy under the timeless perspective (Dennis, 2010; Sauer, 2010a,b). Thus, policymakers concerned with social welfare from date t_0 forth may have incentive to abandon previous commitments.

Computations to solve time-inconsistency problems are difficult because analysis must handle dynamic incentive constraints, which defy Bellman's principle of optimality. Marcet and Marimon (1994, revised in 1998 and 2011) develop a Lagrange method—the recursive saddle point method—to consider incentive constraints in a dynamic economy. Kehoe and Perri (2002) apply the method in a two-country model with incomplete markets. Similar to this study, they numerically solve for the policy function using a version of policy function iteration method closely related to methods for handling occasionally binding constraints (e.g., Christiano and Fisher, 2000) because incentive constraints are occasionally binding. Adam and Billi (2006) studied the optimal commitment policy with an occasionally binding zero lower bound (ZLB) on nominal interest rates utilizing Marcet and Marimon's method.

This reminder of the paper proceeds as follows. Section 2 explains the model, the policy game, and computational procedures and calibrations. Section 3 discusses quantitative properties of optimal sustainable monetary policies, such as impulse responses and stochastic simulations. Section 4 presents an analysis using the alternative parameter sets in Kurozumi (2008) for comparison. Section 5 concludes. Analytical results and computational details appear in the Appendix.

2 Framework for the analysis of optimal monetary policies

Section 2.1 presents a monetary model featuring a stabilization bias and demonstrates the time-inconsistency problem.[4] Section 2.2 reproduces some results in Kurozumi (2008). The optimal sustainable policy is a strategy for the best sustainable equilibrium. It is conducted by following the optimal quasi-sustainable policy derived from the Lagrange method

[4] The model here has a micro-founded structure. See Galí (2008) and Woodford (2003) for its details.

of Marcet and Marimon (1994, revised in 1998 and 2011). Sections 2.3 shows the computational procedure for solving for the policy function of the optimal quasi-sustainable policy, as in Kehoe and Perri (2002). Section 2.4 explains the calibration.

2.1 A monetary model with stabilization bias

The economy contains an infinite number of private agents and a monetary policymaker. Private agents set prices under monopolistic competition in a staggered price setting. Up to the first-order approximation, their behavior is represented by the Phillips curve:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t, \tag{1}$$

for all $t \geq 0$, where π_t is current inflation, $E_t\pi_{t+1}$ is expected future inflation, x_t is the output gap, and u_t is the exogenous price markup shock. $\beta \in (0,1)$ is the discount factor and $\kappa > 0$ is the elasticity of inflation to the output gap. The markup shock u_t follows an AR(1) process:

$$u_t = \rho u_{t-1} + \varepsilon_t, \tag{2}$$

for all $t \geq 0$ and given $u_{-1} = 0$, where $\rho \in (-1,1)$ is persistence of the markup shock, and ε_t is disturbance to the shock with a variance $\sigma_\varepsilon^2 > 0$. ε_t is assumed to be bounded per Kurozumi (2008) and Loisel (2008). The policymaker chooses a sequence of inflation rates and output gaps $\{\pi_t, x_t\}_{t=0}^{\infty}$ to maximize social welfare. That is, the second-order approximation of the representative household's expected utility in Period 0 is

$$V_0 \equiv -E_0 \sum_{t=0}^{\infty} \beta^t [\pi_t^2 + \lambda (x_t - x^*)^2], \tag{3}$$

where $\lambda > 0$ is a weight to stabilize the output gap relative to inflation and x^* is the target level of the output gap, which stems from monopolistic competition and distortion in the steady state. Subsidies are assumed to offset the monopolistic distortion so that the steady state is efficient ($x^* = 0$). Therefore, there is no inflation bias, and stabilization bias is the sole source of time-inconsistency.

The literature usually considers two types of optimal policies for analyzing optimal monetary policy in NK models. Under the optimal discretionary policy, the economy follows a Markov equilibrium in which the state variable is the only markup shock. Policymakers maximize (3) subject to Equation (1) for all $t \geq 0$, given expected future inflation $E_t\pi_{t+1}$.

When $x^* = 0$, the optimal discretionary policy is characterized by the optimality condition

$$\pi_t = -(\lambda/\kappa)x_t, \tag{4}$$

for all $t \geq 0$. In this instance, Equations (1) and (4) determine inflation and the output gap given the exogenous shock process (2). Under the optimal commitment policy, the policymaker can commit to future policies and influence private agents' expectations. The optimal commitment policy is characterized by the optimality condition

$$\pi_t = -(\lambda/\kappa)(x_t - x_{t-1}), \tag{5}$$

for all $t \geq 0$ with an initial condition x_{-1}. Equations (1) and (5) then determine inflation and the output gap given the state variable x_{t-1} and exogenous shock process (2).

Figure 1 graphs how inflation and the output gap are determined under the optimal commitment policy with the initial condition $x_{-1} = 0$ and under the optimal discretionary policy. It demonstrates the time inconsistency of the optimal commitment policy.[5] Inflation and the output gap are determined at the intersection of the upward-sloping Phillips curve and the downward sloping optimality condition. A positive markup shock occurs in Period 0, shifting the Phillips curve upward and inducing a rise in inflation and a drop in the output gap. Under the optimal discretionary policy, given no future inflation ($E_0\pi_1 = 0$), there is a trade-off between stabilizing inflation and stabilizing the output gap. Both are determined at D_0. Under the optimal commitment policy, the policymaker can commit to future deflation in Period 1 ($E_0\pi_1 < 0$), which shifts the Phillips curve downward and accommodates this trade-off by offsetting the markup shock. Inflation and the output gap are determined at C_0, which surpasses D_0 in social welfare. However, the optimality condition shifts downward in Period 1 because of the commitment made in Period 0. Therefore, inflation and the output gap are determined at C_1, and it is worse than at D_1. If the markup shock in Period 1 is absent, the policymaker has incentive to abandon previous commitments and to stabilize both inflation and the output gap in Period 1.

Two types of commitment policies pertain to the lagged output gap in Period 0, x_{-1}. In the *time-0 perspective*, the initial period is period 0 and $x_{-1} = 0$ because no commitment is made in Period 0. In the *timeless perspective*, the initial period is in the indeterminate past, and the initial condition x_{-1} is predetermined. In the timeless perspective, even in Period 0, the policymaker may have incentive to deviate from previous commitments and by doing so can exploit private agents' inflation expectations. The optimal discretionary policy can be superior to the optimal commitment policy under the timeless perspective (Dennis, 2010;

[5] A similar figure appears in Woodford (1999).

Sauer, 2010a,b), even in Period 0, because the initial condition is predetermined.

[Figure 1 is inserted here]

2.2 Optimal sustainable policy

To overcome the time-inconsistency in the optimal commitment policy, Subsection 2.2 considers a policy game between an infinite number of private agents and a policymaker.

2.2.1 Sustainable equilibrium and the sustainability constraint

The policymaker acts first, after which private agents respond to its policy. The history relevant to agents' behavior is given by $h_0 = u_0$ and $h_t = (h_{t-1}, x_{t-1}, u_t)$ for all $t > 0$.[6] The policymaker's strategies are given by $x_t = \sigma_t(h_t)$ and contingent plans for any possible future histories $(\sigma_s)_{s \geq t+1}$. Given current history (h_t, x_t) and $(\sigma_s)_{s \geq t+1}$, private agents' reaction functions are given by $\pi_t = f_t(h_t, x_t)$ and contingent plans for any possible future histories $(f_s)_{s \geq t+1}$. Sustainable equilibrium of the model in the previous section is defined as follows:

Definition 1. A sustainable equilibrium of the model is a pair (σ, f) of a monetary policy strategy and private agents' reaction to the policy strategy such that

(i) given the policy strategy σ and current history (h_t, x_t), the continuation of private agents' reaction f satisfies

$$\begin{aligned} f_t(h_t, x_t) &= \beta E_t[f_{t+1}(h_{t+1}, \sigma_{t+1}(h_{t+1}))] + \kappa x_t + u_t, & t \geq 0 \\ f_s(h_s, \sigma_s(h_s)) &= \beta E_s[f_{s+1}(h_{s+1}, \sigma_{s+1}(h_{s+1}))] + \kappa \sigma_s(h_s) + u_s, & s \geq t+1 \end{aligned}$$

for all possible future histories induced by σ.

(ii) given private agents' reaction f and and current history h_t, the continuation of the policy strategy σ solves

$$\begin{aligned} \max_{(\tilde{\sigma}_s)_{s \geq t}} \quad & -E_t \sum_{s=t}^{\infty} \beta^{s-t} [[f_s(h_s, \tilde{\sigma}_s(h_s))]^2 + \lambda [\tilde{\sigma}_s(h_s)]^2] \\ \text{s.t.} \quad & f_s(h_s, \tilde{\sigma}_s(h_s)) = \beta E_s[f_{s+1}(h_{s+1}, \tilde{\sigma}_{s+1}(h_{s+1}))] + \kappa \tilde{\sigma}_s(h_s) + u_s \end{aligned}$$

for all possible future histories induced by $(\tilde{\sigma}_s)_{s \geq t}$.

[6] Note that π_t is excluded from the public history because private agents are policy takers and only the policymaker may have incentive to deviate from current policy, as in Chari and Kehoe (1990) and Kurozumi (2008).

Under *any* sustainable equilibrium, the policymaker acts so that *the sustainability constraint*,

$$
- E_t \sum_{s=t}^{\infty} \beta^{s-t}(\pi_s^2 + \lambda x_s^2) \geq W^d(u_t), \tag{6}
$$

holds in every period $t \geq 0$, where $W^d(u_t) = -\frac{1+\kappa^2/\lambda}{(1-\beta\rho+\kappa^2/\lambda)^2(1-\beta\rho^2)}\left(u_t^2 + \frac{\beta\sigma_\varepsilon^2}{1-\beta}\right) < 0$ is the worst sustainable equilibrium outcome, which is induced by the optimal discretionary policy.[7] Kurozumi (2008) proves the following propositions:

Proposition 1. *The rational expectations equilibrium (REE) under discretionary policy is the worst sustainable equilibrium of the model.*

Proposition 1 shows that $W^d(u_t)$ attains the lowest value among outcomes of sustainable equilibria. Given the sustainability constraint, the worst outcome assures that the best sustainable equilibrium is included in the set of all sustainable equilibria.

Proposition 2. *Any arbitrary pair (π, x) of contingent sequences of inflation rates and output gaps is an outcome of a sustainable equilibrium if and only if; (i) the pair (π, x) satisfies (1) in every period $t \geq 0$ and (ii) the inequality (6) holds in every period $t \geq 0$.*

Proposition 2 shows that any arbitrary sequence $\{\pi_t, x_t\}_{t=0}^{\infty}$ is an outcome of a sustainable equilibrium if and only if Equations (1) and (6) are satisfied for all $t \geq 0$. These constraints define the entire set of sustainable equilibrium outcomes.

The optimal sustainable policy is a strategy for the best sustainable equilibrium. To analyze it, Kurozumi (2008) derives the optimal quasi-sustainable policy, which induces the best sustainable equilibrium outcome in the *presence* of commitment technologies. The policymaker chooses $\{\pi_t, x_t\}_{t=0}^{\infty}$ to maximize (3) subject to Equations (1) and (6) for $t \geq 0$. Kurozumi argues that "the optimal sustainable policy now becomes a policy strategy which specifies to continue the optimal quasi-sustainable policy as long as it has been adopted in the past; otherwise, the strategy specifies to switch to the optimal discretionary policy forever. This implies that the optimal sustainable policy is conducted by following the optimal quasi-sustainable policy and yields the best sustainable equilibrium outcome in the *absence* of commitment technologies (Kurozumi, 2008, p. 1278)."

If the policymaker abandons its previous policy, the trigger strategy specifies switching to the optimal discretionary policy, which is off the equilibrium path, indefinitely. One might wonder if other equilibria would be chosen if the trigger strategy involved milder punishment—for example, the finite duration in Loisel (2008). Since no theory currently

[7] See Appendix A.1 for the derivation of $W^d(u_t)$. There are no temporary gains from deviating, as in a model with inflationary bias (Ireland, 1997), because the policymaker acts first.

suggests how other equilibria are chosen, it is assumed that the policymaker and private agents switch to the optimal discretionary policy indefinitely, rather than to a less punishing policy.

2.2.2 Characterizing the optimal sustainable policy

To characterize the optimal sustainable policy, this study uses the Lagrange method of Marcet and Marimon to derive the optimal quasi-sustainable policy. The Lagrangian in Period 0 is set up as

$$\begin{aligned} \mathcal{L}_0 \quad \equiv \quad & E_0 \sum_{t=0}^{\infty} \beta^t \left\{ -(\pi_t^2 + \lambda x_t^2) + 2\phi_t(\pi_t - \beta E_t \pi_{t+1} - \kappa x_t - u_t) \right. \\ & \left. -\varphi_t \left[E_t \sum_{s=t}^{\infty} \beta^{s-t} (\pi_s^2 + \lambda x_s^2) + W^d(u_t) \right] \right\}, \end{aligned} \tag{7}$$

where ϕ_t and φ_t are Lagrange multipliers on each constraint (1) and (6) for $t \geq 0$. Abel's summation formula is used with the law of iterated expectations $E_0 \sum_{t=0}^{\infty} \beta^t \varphi_t \left[E_t \sum_{s=t}^{\infty} \beta^{s-t} (\pi_s^2 + \lambda x_s^2) \right] = E_0 \sum_{t=0}^{\infty} \beta^t \left(\sum_{i=0}^{t} \varphi_i \right) (\pi_t^2 + \lambda x_t^2)$ to eliminate the infinite sum in the sustainability constraints, and $E_0 \sum_{t=0}^{\infty} \beta^t \phi_t (\pi_t - \beta E_t \pi_{t+1}) = E_0 \sum_{t=0}^{\infty} \beta^t (\phi_t - \phi_{t-1}) \pi_t$ to eliminate the forward term of the inflation rate:

$$\mathcal{L}_0 \quad = \quad E_0 \sum_{t=0}^{\infty} \beta^t \left\{ -\Psi_t (\pi_t^2 + \lambda x_t^2) + 2(\phi_t - \phi_{t-1}) \pi_t - 2\phi_t (\kappa x_t + u_t) - \varphi_t W^d(u_t) \right\},$$

where $\Psi_t = \Psi_{t-1} + \varphi_t$ with an initial condition $\Psi_{-1} = 1$ and $\phi_{-1} = 0$.[8] From the first order conditions of π_t and x_t, the optimality condition of the optimal quasi-sustainable policy is

$$\pi_t = -\frac{\lambda}{\kappa} \left(x_t - \frac{\Psi_{t-1}}{\Psi_t} x_{t-1} \right), \tag{8}$$

for all $t \geq 0$ with an initial condition x_{-1}. This is different from the optimality conditions of the optimal commitment policy (5) and the optimal discretionary policy (4). In the optimal quasi-sustainable policy (8), the lagged output gap x_{t-1} also appears in the optimality conditions, and the ratio of the sum of Lagrange multipliers on the sustainability constraint $\Psi_{t-1}/\Psi_t = \Psi_{t-1}/(\Psi_{t-1} + \varphi_t) \in (0, 1]$ appears as a coefficient on the lagged output gap.

If the sustainability constraint is slack, the optimal quasi-sustainable policy appears identical to the optimal commitment policy (see Equation (5) and Equation (8) with $\Psi_{t-1}/\Psi_t = 1$). If the sustainability constraint is binding, $\varphi_t > 0$, then $0 < \Psi_{t-1}/\Psi_t < 1$ and the optimal

[8] The Lagrangean corresponds to the saddle point problem in a recursive form considered in Marcet and Marimon (2011).

quasi-sustainable policy is intermediate between the optimal commitment policy and the optimal discretionary policy (see Equations (5) and (4)). Under the optimal quasi-sustainable policy, the policymaker endogenously reduces the steadfastness of its commitment via the lagged output gap, if the current period sustainability constraint is binding.[9]

2.3 Solving for the optimal quasi-sustainable policy

In the optimal quasi-sustainable policy, the equilibrium conditions in period t are summarized as

$$\begin{aligned}
x_t &= -\frac{\kappa}{\lambda}\pi_t + \frac{\Psi_{t-1}}{\Psi_t}x_{t-1},\\
\Psi_t &= \Psi_{t-1} + \varphi_t,\\
\pi_t &= \beta E_t \pi_{t+1} + \kappa x_t + u_t,\\
-E_t &\sum_{s=t}^{\infty} \beta^{s-t}(\pi_s^2 + \lambda x_s^2) \geq W^d(u_t),
\end{aligned}$$

for $t \geq 0$, given initial conditions x_{-1} and $\Psi_{-1} = 1$. This is the four-equation system of four endogenous variables $\{x_t, \pi_t, \Psi_t, \varphi_t\}_{t=0}^{\infty}$ given a sequence of exogenous shock $\{u_t\}_{t=0}^{\infty}$. Therefore, the system can be solved in principle, but not via the usual methods because there is an infinite sum of future variables in the sustainability constraints, which occasionally bind given the exogenous stochastic process. Instead, a version of the policy function iteration method is used, as in Kehoe and Perri (2002).[10] To this end, the value of continuing the current policy is defined as $V_t \equiv -E_t \sum_{s=t}^{\infty} \beta^{s-t}(\pi_s^2 + \lambda x_s^2)$ (Note that the policymaker maximizes social welfare in Period 0, V_0). The system is written recursively as

$$\begin{aligned}
V_t &= -(\pi_t^2 + \lambda x_t^2) + \beta E_t V_{t+1},\\
x_t &= -\frac{\kappa}{\lambda}\pi_t + z_t x_{t-1},\\
\pi_t &= \kappa x_t + u_t + \beta E_t \pi_{t+1},\\
V_t &\geq W^d(u_t),
\end{aligned}$$

where $z_t \equiv \Psi_{t-1}/\Psi_t \in (0, 1]$ is the steadfastness of commitment, which equals 1 if the sustainability constraint is slack, and less than 1 if it is binding. u_t and x_{t-1} are the state

[9] One may ask whether the binding constraint imposes a cost on the policymaker that might increase its incentive to abandon the optimal quasi-sustainable policy. The question is moot, as the presence of commitment technologies is assumed.

[10] In Kehoe and Perri (2002), the worst sustainable equilibrium value is a function of the endogenous state variable. In this study, it is a function only of the exogenous state variable.

variables. V_t, π_t, x_t and z_t are the jump variables.[11] Let $s = (u, x_{-1}) \in U \times X$ where U and X are closed sets. A state-space representation is then obtained:

$$\begin{aligned}
V(s) &= -([\pi(s)]^2 + \lambda[x(s)]^2) + \beta \sum_{u'} p(u'|u)V(u', x(s)), \\
x(s) &= -\frac{\kappa}{\lambda}\pi(s) + z(s)x_{-1}, \\
\pi(s) &= \kappa x(s) + u + \beta \sum_{u'} p(u'|u)\pi(u', x(s)), \\
V(s) &\geq W^d(u),
\end{aligned}$$

where $\pi(s)$, $x(s)$, and $z(s)$ are the policy functions and $V(s)$ is the value function. $p(u'|u)$ represents a transition probability matrix, which approximates the AR(1) process in Equation (2). This system has a recursive structure with regard to $V(s)$ and $\pi(s)$, making it possible to apply a policy function iteration method to the system. The occasionally binding constraint $V(s) \geq W^d(u)$ must be addressed; hence, two cases are considered at each grid point $s \in U \times X$. The constraint is slack or binding. Appendix A.2 provides the computational details.

This approach characterizes the behavior of the optimal quasi-sustainable policy directly and solves the equilibrium conditions for the policy functions instead of solving the sustainability constraint for the lower bound of the discount factor $\underline{\beta}$, as explained in Section 4. Then the approach calculates the probabilities of binding constraints (and other statistics), unlike Kurozumi. These results are quantitative; Kurozumi's results are qualitative, as he checked only whether the optimal commitment policy is sustainable.

2.4 Calibration

Parameter values $(\beta, \kappa, \lambda, \rho, \sigma_\varepsilon)$ are calibrated, and the ranges of state spaces U and X are set. Table 1 summarizes the parameter values. For the slope of the Phillips curve and the weight to the output gap, $\kappa = 0.024$ and $\lambda = 0.048/4^2 = 0.003$ are chosen. These were originally estimated by Rotemberg and Woodford (1998) and Woodford (2003) and are common in the literature (Adam and Billi, 2006; 2007; Nakov, 2008; Giannoni, 2010; Bodenstein, Hebden, and Nunes (2012)).[12] Estimates by Adam and Billi (2006; 2007) serve as markup shock parameters; $\rho = 0.0$ and $\sigma_\varepsilon = 0.154$ based on quarterly U.S. data from 1983:1 to 2002:4. Finally, the discount factor to match a 3.5% annual real interest rate is calibrated; it implies

[11] $z_t = \Psi_{t-1}/(\Psi_{t-1} + \varphi_t)$ is a jump variable because it depends on the Lagrange multiplier in the current period φ_t.

[12] Note that π_t is the quarterly inflation rate in the model, whereas Woodford estimates using an annual inflation rate. Therefore, I divide the original estimated value of λ by 4^2.

that $\beta = 1/(1+\frac{3.5\%}{4}) = 0.9913$. $\beta = 0.9$ is used to demonstrate the difference between the optimal quasi-sustainable policy and the two other. Several alternative parameters sets are used to compare quantitative results with Kurozumi's qualitative results in Section 4. The upper bound of the disturbance is $m\sigma_\varepsilon$ and $m = 6$ is set.[13] Maximum and minimum values of X are chosen to be consistent with the optimal volatility of the output gap. The number of grids is $n_u = 31$ for U and $n_x = 15$ for X. Tauchen's (1986) method is used to approximate the AR(1) process (Equation (2)) with parameters $(\rho, \sigma_\varepsilon, m)$ for U and $p(u'|u)$. Use of the Tauchen approximation makes $u_t \in U$ virtually bounded, as in Kurozumi (2008) and Loisel (2008). For comparison, the policy and value functions under the optimal commitment and discretionary policies are numerically obtained by policy function iteration method.

[Table 1 is inserted here]

3 Numerical results

This section shows impulse response functions of disturbances to the markup shock and stochastic simulations. Section 3.1 demonstrates differences among the optimal quasi-sustainable policy, the optimal time-0 commitment policy (i.e., the optimal commitment policy under the time-0 perspective), and the optimal discretionary policy with a low discount factor $\beta = 0.9$ and a large shock. The optimal quasi-sustainable policy is intermediate between the optimal time-0 commitment policy and the optimal discretionary policy. By contrast, in Section 3.2, using the high discount factor $\beta = 0.9913$ common in the literature, the difference between the optimal quasi-sustainable policy and the optimal commitment policy is *practically* zero. This result contrasts with Kurozumi's result with the upper bound of the markup shock and lagged output gap.

3.1 Impulse responses and policy functions: the case with $\beta = 0.9$

Figure 2 computes the impulse response functions of a large disturbance ε_0 ($5 \times 0.154\% = 0.77\%$), with a low discount factor $\beta = 0.9$ and initial condition $x_{-1} = 0$, under the time-0 perspective. The chosen discount factor $\beta = 0.9$ is extremely low for a quarterly model. Therefore, the policymaker gives less consideration to a long-term relationship with private agents and the optimal commitment policy is less likely to be sustainable. Stochastic simulations with a high discount factor and reasonably-sized shocks appear in the next subsection.

Under the optimal discretionary policy, the economy responds in Period 0 only; as there is no persistence in the markup shock ($\rho = 0$), inflation rises and the output gap drops. Under

[13] This is larger than $m = 3$, the value commonly used in the literature (Judd, 1992).

the optimal time-0 commitment policy, the policymaker promises deflation after Period 1 to accommodate a trade-off between stabilizing inflation and the output gap in Period 0.

Under the optimal quasi-sustainable policy, the policymaker promises less deflation after Period 1 and less stabilization in Period 0 than under the optimal time-0 commitment policy. Period 1 features an incentive for the policymaker to deviate from the past commitment, and the optimal quasi-sustainable policy diminishes the steadfastness of the policymaker's commitment. The sustainability constraint in Period 1 is binding ($\varphi_t > 0$), and the ratio of the sum of Lagrange multipliers on the sustainability constraint Ψ_{t-1}/Ψ_t takes a value below 1.

[Figure 2 is inserted here]

The optimal quasi-sustainable policy falls between the optimal time-0 commitment policy and the optimal discretionary policy. To inspect the mechanism behind this result, the policy function of $z_t(u_t, x_{t-1}) = \Psi_{t-1}/\Psi_t$ is examined in Figure 3.a. In Period 0, the markup shock rises ($u_0 > 0$) and the lagged output gap is zero ($x_{-1} = 0$). Therefore, the value of continuing the current policy $V(u_0, x_{-1})$ exceeds the value of switching to the optimal discretionary policy $W^d(u_0)$, and the sustainability constraint in Period 0 is slack ($V(u_0, x_{-1}) > W^d(u_0)$). In Period 1, the markup shock reverts to the mean ($u_1 = 0$), but the lagged output gap remains negative ($x_0 < 0$). Therefore, $V(u_1, x_0)$ is lower than $W^d(u_1)$, and the sustainability constraint in Period 1 is binding ($V(u_1, x_0) = W^d(u_1)$). This makes the coefficient on the lagged output gap less than 1 ($z(u_1, x_0) = \Psi_0/\Psi_1 < 1$).

The difference between the conditional expected utility induced by the optimal timeless commitment policy $V^c(u_t, x_{t-1})$ and the conditional expected utility induced by the optimal discretionary policy $W^d(u_t)$ is analytically obtained (Figure 3.b).[14] $-\max\{V^c(u_t, x_{t-1})/W^d(u_t) - 1, 0\}$ is used for a measure of the difference. If $V^c(u_t, x_{t-1}) < W^d(u_t)$, it takes a value less than 0.

Figure 3.a shows the region wherein the sustainability constraint is binding and the policy function of $z(u_t, x_{t-1})$ takes a value less than 1. It largely corresponds to the region where the optimal discretionary policy is superior to the optimal timeless commitment policy in terms of the social welfare in Period t (Figure 3.b). When the absolute value of u_t is smaller and/or the absolute value of x_{t-1} is larger, $z(u_t, x_{t-1}) < 1$ and $V^c(u_t, x_{t-1}) < W^d(u_t)$ are more likely to hold. This is because the value of policymakers' commitment is lower—the outside option is better—when u_t is smaller, whereas there is more incentive for the policymaker to abandon the current policy when x_{t-1} is larger.

[14] See also Appendix A.1. Sauer (2010a) shows a similar figure to argue that the optimal discretionary policy can be superior to the optimal commitment policy under the timeless perspective, when the economy is currently far from the steady state.

[Figure 3 is inserted here]

Note that the optimal quasi-sustainable policy is also intermediate between the optimal time-0 commitment policy and the optimal discretionary policy, even in Period 0 when the sustainability constraint is slack. In Figure 4, the policy functions of $\pi(u_t, 0)$ and $x(u_t, 0)$ are plotted. When the lagged output gap x_{t-1} equals 0, the sustainability constraint in Period t is slack ($z(u_t, 0) = 1$) and $V^c(u_t, 0) > W^d(u_t)$ holds for every $u_t \in U$, as Figure 3 shows. When the markup shock is large, however, the optimal quasi-sustainable policy differs from the optimal time-0 commitment policy and the optimal discretionary policy. This occurs because rational expectations take into account the future possibility of binding constraints. The markup shock reduces the output gap in Period 0, and the sustainability constraint to bind via the dropped lagged output gap in Period 1. Thus, the policymaker weakens stabilizing inflation and the output gap in Period 0.

[Figure 4 is inserted here]

3.2 Stochastic simulations

Stochastic simulations with a reasonably sized markup shock examine how the time-inconsistency problem is relevant to actual conduct of monetary policy. The same exercise is performed for each of the optimal quasi-sustainable, commitment, and discretionary policies using the numerically obtained policy functions.

Using the sequence obtained by stochastic simulations of the policy functions, the probability of binding sustainability constraint $Pr\{z < 1\}$ is calculated, which exceeds 0 when the optimal commitment policy is not sustainable. The unconditional expected (discounted sum of) utility $\bar{V} = EV(x_{t-1}, u_t) = (1-\beta)^{-1}(\sigma_\pi^2 + \lambda\sigma_x^2)$ is also calculated, where σ_π^2 and σ_x^2 are the unconditional variances of inflation and the output gap. The conditional expected utility $V(0, 0)$ is obtained by evaluating the value function at $(u, x_{-1}) = (0, 0)$.

The optimal time-0 commitment policy always surpasses the conditional expected utility of the optimal quasi-sustainable policy because the policymaker maximizes the conditional expected utility itself in Period 0 given $(u, x_{-1}) = (0, 0)$. A less constrained problem without the sustainability constraint is solved for the optimal time-0 commitment policy than the problem with the constraint for the optimal quasi-sustainable policy. By contrast, the unconditional expected utility of the optimal quasi-sustainable policy can surpass that of the optimal timeless commitment policy, for it reflects the variation of the lagged output gap in the initial period and diminishes the steadfastness of commitment via the lagged output gap.[15]

The result of stochastic simulations appears in Table 2. When $\beta = 0.9$, the probability of binding constraints $Pr\{z < 1\}$ is approximately 6.8%. This figure seems small (once in five years in a quarterly model), but it slightly affects the optimal volatilities of inflation and the output gap (σ_π^2 and σ_x^2) and hence unconditional expected utility ($\bar{V} = (1-\beta)^{-1}(\sigma_\pi^2 + \lambda\sigma_x^2)$). The optimal quasi-sustainable policy yields higher social welfare than the optimal timeless commitment policy with regard to unconditional expected utility because it diminishes the steadfastness of commitment and reduces volatility of inflation and the output gap.[16]

However, when $\beta = 0.9913$, the difference between the optimal quasi-sustainable policy and the optimal commitment policy is *practically* zero in terms of conditional or unconditional expected utility.[17] In other words, policymakers have no incentive to deviate from the Ramsey-optimal commitment policy under the parameter set commonly used in the literature.[18]

[Table 2 is inserted here]

4 Difference from Kurozumi (2008)

The framework presented in Sections 2.1 and 2.2 originates in Kurozumi (2008). This study uses a method different from Kurozumi, inducing differences in the simulation result. This section explains such differences.

To evaluate the sustainability constraint, Kurozumi (2008) assumes that disturbance ε is bounded by $m\sigma_\varepsilon$ where m is an integer, and the upper bounds on the markup shock and lagged output gap of the optimal commitment policy are $u_t \leq B = m\sigma_\varepsilon/(1+|\rho|)$ and $x_{t-1} \leq |a_x|B/(1-b_x)$, where a_x and b_x are the coefficients of the policy function of the output gap ($x_t = a_x u_t + b_x x_{t-1}$). Then Kurozumi calculates a lower bound on the discount

[15] This is consistent with the argument that the optimal discretionary policy can be superior to the optimal timeless commitment policy when the lagged output gap is large (Dennis, 2010; Sauer, 2010a,b).

[16] Jensen and McCallum (2002) point out that other policy rules are better than the optimal timeless commitment policy to maximize conditional or unconditional expected utility. However, rules proposed in Jensen and McCallum (2002) are not time-consistent, whereas the optimal quasi-sustainable policy is.

[17] This result is only up to a precision of numerical solutions. Approximation errors are very small especially when $\beta = 0.9913$, which are computed in Appendix A.2.

[18] As Section 2.2.1 explained, it is assumed that if the policymaker abandons the previously adopted policy, he/she and all private agents switch to the optimal discretionary policy indefinitely. If other equilibria were chosen with a less harsh trigger strategy, the policymaker may have incentive to deviate from the Ramsey-optimal commitment policy.

factor for which the optimal commitment policy is sustainable. This is given by[19]

$$\underline{\beta} = \inf\{\beta \in (0,1) : F(\tilde{\beta}) \geq 0 \forall \tilde{\beta} \in (\beta, 1)\},$$

where

$$F(\beta) = V^c(B, |a_x|B/(1-b_x)) - W^d(B).$$

This approach, however, can find only whether the sustainability constraint is binding for the maximum size of the markup shock and a particular value of β. The policy function approach solves for the policy function instead, and calculates the probability of a binding constraint with a reasonably sized markup shock.

Table 3.a shows the probability of binding constraints for parameter values (κ, λ, ρ) considered in Kurozumi (2008).[20] The probability of a binding constraint is 0 except when λ/κ is high. The optimal commitment policy is sustainable for some plausible parameters, including the baseline parameters in Section 3. This finding contrasts with Kurozumi's finding. In Table 2 of Kurozumi (2008), the optimal commitment policy is not sustainable with $\beta = 0.99$—i.e., $\beta = 0.99 < \underline{\beta}$ for most parameter values of (κ, λ, ρ).[21] Also, Table 3.a shows that the optimal commitment policy is less likely to be sustainable when λ/κ or ρ is higher, i.e., the probability of binding constraints is higher. This finding coincides with Kurozumi's finding for λ/κ, but not for ρ.

The variation of the lagged output gap, which Kurozumi's analysis disregards, is the key to understand the differences. As Section 3.1 and Figure 3 explain, two opposite forces threaten the sustainability of the optimal commitment policy. First, the smaller the absolute value of u_t is, the better is the outside option of the optimal discretionary policy, and policymakers have greater incentive to abandon their current policy. Second, the larger the absolute value of x_{t-1} is, the more likely policymakers are to abandon its previous commitment. When ρ is higher, u_t is larger, which also increases the lagged output gap in the next period x_t. The effect of the latter dominates that of the former. Therefore, the higher the ρ, the less likely is the optimal commitment policy sustained. Kurozumi disregards the latter effect—the variation of the lagged output gap.

Table 3.b compares the ratio of the unconditional mean of the output gap under the optimal commitment policy to that under the optimal discretionary policy (σ_x^c/σ_x^d), which

[19] The difference in social welfare between the optimal commitment and discretionary policies $V^c(u_t, x_{t-1}) - W^d(u_t)$ is analytically obtained in Appendix A.1.

[20] The other parameters are fixed $(\beta, \sigma_\varepsilon, m) = (0.99, 0.154, 6)$. The standard deviation σ_ε is irrelevant to the probability of a binding constraint because we can rewrite $F(\beta) = \sigma_\varepsilon^2 \tilde{F}(\beta)$.

[21] The best Markov quasi-sustainable policy is also calculated in Table 3 of Kurozumi (2008); it is a linear function of the markup shock. This study calculates the optimal quasi-sustainable policy, which is a non-linear function of the markup shock and lagged output gap.

can be calculated analytically. When λ/κ and/or ρ is higher, the output gap is relatively more volatile under the optimal commitment policy. The sustainability constraint is more likely to be binding, and the optimal sustainable policy diminishes the steadfastness of commitment via the lagged output gap.

If the lower bound of discount factor $\underline{\beta}$ is calculated using the method in Kurozumi (2008) but with the unconditional variances of the markup shock and lagged output gap, also used in Loisel (2008), the result obtained is consistent with that using the policy function iteration method (Table 3.c). The higher the ρ, the less likely is the optimal commitment policy sustained (i.e., the higher the $\underline{\beta}$) because the variation of the lagged output gap is taken into account.

[Table 3 is inserted here]

5 Conclusion

This study solved for the policy functions of the optimal quasi-sustainable monetary policy to invite a quantitative analysis of optimal sustainable monetary policies as proposed by Kurozumi (2008). Stochastic simulations show that the difference between the optimal quasi-sustainable policy and the optimal commitment policy is *practically* zero for a parameter set calibrated using U.S. data. In an economy with some alternative parameters, the optimal quasi-sustainable policy yields higher social welfare in terms of unconditional expected utility than the optimal commitment policy. The dynamics of the lagged output gap provide the key to such differences. When the lagged output gap, which measures policymakers' previous commitment, is large, the optimal quasi-sustainable policy endogenously diminishes the steadfastness of policymakers' commitment via the lagged output gap.

One interesting direction for future research is to introduce a ZLB on the nominal interest rate into the model. Following a sizable economic shock, the presence of ZLB is also crucial for conduct of monetary policy.[22] When a ZLB is explicitly considered, markup shocks and real rate shocks affect the short-term dynamics of inflation and the output gap because the ZLB prevents policymakers from controlling the real economy in the short run. For quantitative analysis, the optimization problem is solved with the sustainability constraint and the ZLB, both of them occasionally binding constraints, by applying techniques used in this study.[23]

[22] See Adam and Billi (2006; 2007), Nakov (2008) and Bodenstein, Hebden, and Nunes (2012). Nakata (2014) examines sustainable plans in a NK model with ZLB, but he checks only whether the sustainability constraint is binding, like Kurozumi (2008) and Loisel (2008).

[23] However, the optimal discretionary policy might not induce the worst sustainable equilibrium value, which is on the right-hand side of the sustainability constraint, when ZLB is imposed.

Solving a non-linear version of the model also remains for further research.[24] This study considers the non-linearity induced by the sustainability constraint; hence, considering the original non-linear model is a logical extension of the linear-quadratic framework considered here.

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[24] See Anderson, Kim, and Yun (2010) and Van Zandweghe and Wolman (2010) for the Calvo-type (1983) price adjustment model.

A Appendix

A.1 Analytical solutions

The policy rules for the optimal commitment policy and the optimal discretionary policy are given by:

$$\begin{aligned} x_t^c &= a_x u_t + b_x x_{t-1}^c, \quad & \pi_t^c &= a_\pi u_t + b_\pi x_{t-1}^c, \\ x_t^d &= c_x u_t, & \pi_t^d &= c_\pi u_t, \end{aligned}$$

where

$$\begin{aligned} a_x &= -(\kappa/\lambda)/[\beta(b^+ - \rho)] < 0, \quad & a_\pi &= (1/[\beta(b^+ - \rho)]) > 0, \\ b_x &= b^- \in (0,1), & b_\pi &= (\lambda/\kappa)(1 - b^-) > 0, \\ c_x &= -(\kappa/\lambda)/(1 - \beta\rho + \kappa^2/\lambda) < 0, & c_\pi &= 1/(1 - \beta\rho + \kappa^2/\lambda) > 0, \end{aligned}$$

and $b^\pm = [(1 + \beta + \kappa^2/\lambda) \pm ((1 + \beta + \kappa^2/\lambda)^2 - 4\beta)^{.5}]/(2\beta)$ is the solution of a quadratic equation $f(b) \equiv \beta b^2 - (1 + \beta + \kappa^2/\lambda)b + 1 = 0$. Note that $b^+ > 1$ and $b^- < 1$.

For measures of social welfare, expected utilities conditioned on initial conditions (u_t, x_{t-1}) are given by[25]

$$W^d(u_t) = -\frac{c_\pi^2 + \lambda c_x^2}{1 - \beta\rho^2}\left(u_t^2 + \frac{\beta\sigma_\varepsilon^2}{1-\beta}\right), \tag{9}$$

$$V^c(u_t, x_{t-1}) = -\frac{b_\pi^2 + \lambda b_x^2}{1 - \beta b_x^2} x_{-1}^2 - \left(a_\pi^2 + \lambda a_x^2 + \frac{\beta a_x^2(b_\pi^2 + \lambda b_x^2)}{1 - \beta b_x^2}\right)\frac{1}{1 - \beta\rho^2}\left(u_t^2 + \frac{\beta\sigma_\varepsilon^2}{1-\beta}\right) \tag{10}$$

Also, unconditional variances of the markup shock and output gap are given by

$$\sigma_u^2 = \frac{\sigma_\varepsilon^2}{1 - \rho^2}, \quad (\sigma_x^c)^2 = \frac{a_x^2(1 + b_x\rho)\sigma_\varepsilon^2}{(1 - b_x^2)(1 - b_x\rho)(1 - \rho^2)}, \quad (\sigma_x^d)^2 = c_x^2\sigma_u^2.$$

A.2 Computation

A.2.1 Numerical algorithm

To compute the policy function, a version of the policy function iteration method with occasionally binding constraints is used, as in Kehoe and Perri (2002). Let $s = (u, x_{-1}) \in$

[25] The form of conditional expected utility for the optimal commitment policy is different from Equation (6) in Kurozumi (2008), because $(b_\pi^2 + \lambda b_x^2)(1 - \beta b_x^2)^{-1}\beta a_x b_x + a_\pi b_\pi + \lambda a_x b_x = 0$ is utilized.

$U \times X$ where U and X are closed sets. A state-space representation is written as:

$$
\begin{aligned}
V(s) &= -([\pi(s)]^2 + \lambda[x(s)]^2) + \beta \sum_{u'} p(u'|u)V(u', x(s)), \\
x(s) &= -\frac{\kappa}{\lambda}\pi(s) + z(s)x_{-1}, \\
\pi(s) &= \kappa x(s) + u + \beta \sum_{u'} p(u'|u)\pi(u', x(s)), \\
V(s) &\geq \hat{W}^d(u),
\end{aligned}
$$

where $V(s)$, $\pi(s)$, $x(s)$ and $z(s)$ are the policy functions and $\hat{W}^d(u)$ is the worst sustainable equilibrium value induced by the discretionary policy, which is computed numerically. The algorithm is as follows:

1. Set the initial guess of the functions $V^{(0)}(s)$ and $\pi^{(0)}(s)$ where $s = (u, x_{-1})$ is each grid point on $U \times X$.
2. Solve the relevant equations for the values $(V^{i,s}, \pi^{i,s}, x^{i,s}, z^{i,s})$ for every grid point s, given the functions $V^{(i-1)}(u', x)$ and $\pi^{(i-1)}(u', x)$.
3. Set new functions $V^{(i)}(s) = \{V^{i,s}\}_{s \in U \times X}$ and $\pi^{(i)}(s) = \{\pi^{i,s}\}_{s \in U \times X}$.
4. Iterate 2-3 until the functions $V^{(i)}(s)$ and $\pi^{(i)}(s)$ converge at each grid point.

In Step 2, there are two possible binding patterns of the sustainability constraint: (i) the constraint is slack or (ii) the constraint binds.

(i) $z^{i,s} = 1$. Then I can solve

$$
\begin{aligned}
V^{i,s} &= -([\pi^{i,s}]^2 + \lambda[x^{i,s}]^2) + \beta \sum_{u'} p(u'|u)V^{(i-1)}(u', x^{i,s}), \\
x^{i,s} &= -\frac{\kappa}{\lambda}\pi^{i,s} + x_{-1}, \\
\pi^{i,s} &= \kappa x^{i,s} + u + \beta \sum_{u'} p(u'|u)\pi^{(i-1)}(u', x^{i,s}),
\end{aligned}
$$

for the values of $(x^{i,s}, \pi^{i,s}, V^{i,s})$.

(ii) $z^{i,s} \in (0, 1)$ and $V^{i,s} = \hat{W}^d(u)$. I can solve

$$
\begin{aligned}
\hat{W}^d(u) &= -([\pi^{i,s}]^2 + \lambda[x^{i,s}]^2) + \beta \sum_{u'} p(u'|u)V^{(i-1)}(u', x^{i,s}), \\
x^{i,s} &= -\frac{\kappa}{\lambda}\pi^{i,s} + z^{i,s}x_{-1}, \\
\pi^{i,s} &= \kappa x^{i,s} + u + \beta \sum_{u'} p(u'|u)\pi^{(i-1)}(u', x^{i,s}),
\end{aligned}
$$

for the values of $(x^{i,s}, \pi^{i,s}, z^{i,s})$.

$x^{i,s}$ may not be on the grid points. Therefore, the functions need to be approximated. Spline interpolation is used between the grid points X, and linear outerpolation is used outside of X.[26] More specifically, the conditional expectations are approximated as $h_{vi}(x) = \sum_{u_j} p(u_j|u_i)V(u_j, x)$ and $h_{\pi i}(x) = \sum_{u_j} p(u_j|u_i)\pi(u_j, x)$ for each $i = 1, ..., n_u$, by cubic splines.

The worst sustainable equilibrium value $\hat{W}^d(u)$ is computed easier. From Equations (1) and (4):

$$\left(1 + \frac{\kappa^2}{\lambda}\right)\pi_t = \beta E_t \pi_{t+1} + u_t.$$

It can be written as

$$\left(1 + \frac{\kappa^2}{\lambda}\right)\pi(u_i) = \beta \sum_j p(u_j|u_i)\pi(u_j) + u_i,$$

This is linear in $\pi(u_i)$ for each grid $u_1, u_2, ..., u_{n_u}$. Equivalently,

$$\begin{bmatrix} 1 + \frac{\kappa^2}{\lambda} - \beta p(u_1|u_1) & -\beta p(u_2|u_1) & \cdots & -\beta p(u_{n_u}|u_1) \\ -\beta p(u_1|u_2) & 1 + \frac{\kappa^2}{\lambda} - \beta p(u_2|u_2) & \cdots & -\beta p(u_{n_u}|u_2) \\ \vdots & \vdots & \ddots & \vdots \\ -\beta p(u_1|u_{n_u}) & -\beta p(u_2|u_{n_u}) & \cdots & 1 + \frac{\kappa^2}{\lambda} - \beta p(u_{n_u}|u_{n_u}) \end{bmatrix} \begin{bmatrix} \pi(u_1) \\ \pi(u_2) \\ \vdots \\ \pi(u_{n_u}) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n_u} \end{bmatrix},$$

which can be easily solved for $\pi(u_i)$. Having $\pi(u_i)$, substituting (4) into (3) yields:

$$\hat{W}^d(u_i) = -\left(1 + \frac{\kappa^2}{\lambda}\right)[\pi(u_i)]^2 + \beta \sum_j p(u_j|u_i)\hat{W}^d(u_j),$$

which can be solved similarly for $\hat{W}^d(u_i)$.

A.2.2 Approximation errors

The optimal quasi-sustainable policy is obtained by numerical method, therefore, the accuracy of the solution is needed to be assessed. Following Adam and Billi (2006), the residual function is computed:

$$R(\tilde{s}) = -V(\tilde{s}) - ([\pi(\tilde{s})]^2 + \lambda[x(\tilde{s})]^2) + \beta h_{vi}(x(\tilde{s})),$$

[26]It may be justified by observing that the policy function in the optimal commitment policy is linear, which is obtained by solving the problem without the sustainability constraint.

where $\tilde{s} \in \tilde{X} \times U$ has finer grid points than the original ones to solve for the policy function. A total of 201 grids are used for $\tilde{X}$ compared to the original 15 grids on X[27]. Note that $R(s) = 0$ exactly holds on $s \in X \times U$. Linear interpolation is used to evaluate $V(\tilde{s})$, $\pi(\tilde{s})$, and $x(\tilde{s})$, which are off the original grids. For the baseline parameters, the absolute and relative errors are calculated as $e^{abs} = \|R(\tilde{s})\|_\infty$ and $e^{rel} = \|R(\tilde{s})/V(\tilde{s})\|_\infty$. When $\beta = 0.9$, $e^{abs} = 0.0013$ and $e^{rel} = 0.0079$. When $\beta = 0.9913$, $e^{abs} = 0.0016$ and $e^{rel} = 0.0008$. If the sustainability constraints never bind, the errors are almost zero.

A.3 Further numerical results

Figure 5 shows a sample simulated path with baseline parameters and $\beta = 0.9$. A total of 51 periods (from period 500 to 550) are selected from a simulated sample of 1000 periods, to see the dynamics with reasonably-sized shocks. At period 510, the sustainability constraint is binding. Before that, negative disturbances sequentially hit the markup shock and the output gap rises. After the markup shock reverts to the mean, a remaining large output gap triggers the constraint to bind, and then the output gap in the optimal quasi-sustainable policy drops more than the output gap in the optimal commitment policy does. At periods 505, 546 and 548, the sustainability constraint is also binding, but the effect of binding constraints on the output gap is much smaller.

[Figure 5 is inserted here]

Table 4 looks at the unconditional means of inflation and the output gap, as well as the unconditional and conditional expected utilities. The unconditional means of inflation σ_π is not significantly affected by binding constraints. Those in the optimal quasi-sustainable and commitment policies are almost the same. When λ/κ is higher, the unconditional expected utility $\bar{V}$ in the optimal quasi-sustainable policy is higher than the unconditional expected utility in the optimal commitment and discretionary policies. As predicted, the conditional expected utility $V(0,0)$ takes the highest value in the optimal commitment policy for all parameter sets considered here, and it takes the same value in the optimal quasi-sustainable policy when the probability of binding constraints is 0.

[Table 4 is inserted here]

[27] The same grids are used as the original ones for U because Tauchen's method is used.

References

Abreu, D. (1988): “On the Theory of Infinitely Repeated Games with Discounting,” *Econometrica*, 56(2), 383–396.

Adam, K., and R. M. Billi (2006): “Optimal Monetary Policy under Commitment with a Zero Bound on Nominal Interest Rates,” *Journal of Money, Credit, and Banking*, 38(7), 1877–1905.

——— (2007): “Discretionary Monetary Policy and the Zero Lower Bound on Nominal Interest Rates,” *Journal of Monetary Economics*, 54(3), 728–752.

Anderson, G. S., J. Kim, and T. Yun (2010): “Using a Projection Method to Analyze Inflation Bias in a Micro-Founded Model,” *Journal of Economic Dynamics and Control*, 34(9), 1572–1581.

Barro, R. J., and D. B. Gordon (1983): “A Positive Theory of Monetary Policy in a Natural Rate Model,” *Journal of Political Economy*, 91(4), 589–610.

Bodenstein, M., J. Hebden, and R. Nunes (2012): “Imperfect Credibility and the Zero Lower Bound on the Nominal Interest Rate,” *Journal of Monetary Economics*, (59), 135–149.

Calvo, G. A. (1983): “Staggered Prices in a Utility-Maximizing Framework,” *Journal of Monetary Economics*, 12(3), 383–398.

Chari, V. V., and P. J. Kehoe (1990): “Sustainable Plans,” *Journal of Political Economy*, 98(4), 783–802.

Christiano, L., and J. Fisher (2000): “Algorithms for Solving Dynamic Models with Occasionally Binding Constraints,” *Journal of Economic Dynamics and*, 24(8), 1179–1232.

Dennis, R. (2010): “When Is Discretion Superior to Timeless Perspective Policymaking?,” *Journal of Monetary Economics*, 57(3), 266–277.

Galí, J. (2008): *Monetary Policy, Inflation, and the Business Cycle*. Princeton University Press, Princeton, NJ.

Giannoni, M. P. (2010): “Optimal Interest-Rate Rules in a Forward-Looking Model, and Inflation Stabilization versus Price-Level Stabilization,” NBER Working Paper 15986.

Ireland, P. (1997): “Sustainable Monetary Policies,” *Journal of Economic Dynamics and Control*, 22(1), 87–108.

Jensen, C., and B. T. McCallum (2002): "The Non-Optimality of Proposed Monetary Policy Rules under Timeless-Perspective Commitment," *Economics Letters*, 77, 163–168.

Judd, K. L. (1992): "Projection Methods for Solving Aggregate Growth Models," *Journal of Economic Theory*, 58(2), 410–452.

Kehoe, P. J., and F. Perri (2002): "International Business Cycles with Endogenous Incomplete Markets," *Econometrica*, 70(3), 907–928.

Kurozumi, T. (2008): "Optimal Sustainable Monetary Policy," *Journal of Monetary Economics*, 55(7), 1277–1289.

Kydland, F. E., and E. C. Prescott (1977): "Rules Rather than Discretion: The Inconsistency of Optimal Plans," *Journal of Political Economy*, 85(3), 473–491.

Loisel, O. (2008): "Central bank reputation in a forward-looking model," *Journal of Economic Dynamics and Control*, 32(11), 3718–3742.

Marcet, A., and R. Marimon (1994): "Recursive Contracts," Economics Working Papers 337, Department of Economics and Business, Universitat Pompeu Fabra.

——— (1998): "Recursive Contracts," EUI Working Papers, ECO 1998/37.

——— (2011): "Recursive Contracts," EUI Working Papers, ECO 2011/15.

Nakata, T. (2014): "Reputation and Liquidity Traps," FEDS Working Paper 2014-50, Board of Governors of the Federal Reserve System.

Nakov, A. (2008): "Optimal and Simple Monetary Policy Rules with Zero Floor on the Nominal Interest Rate," *Journal of International Central Banking*, 4(2), 73–127.

Rotemberg, J., and M. Woodford (1998): "An Optimization-Based Econometric Framework for the Evaluation of Monetary Policy," *NBER Macroeconomics Annual 1997*, 12, 297–361.

Sauer, S. (2010a): "Discretion Rather Than Rules? When Is Discretionary Policymaking Better Than the Timeless Perspective?," *Journal of International Central Banking*, 6(2), 1–29.

Sauer, S. (2010b): "When Discretion Is Better: Initial Conditions and the Timeless Perspective," *Economics Letters*, 107(2), 128–130.

TAUCHEN, G. (1986): "Finite State Markov-Chain Approximations to Univariate and Vector Autoregressions," *Economics Letters*, 20(2), 177–181.

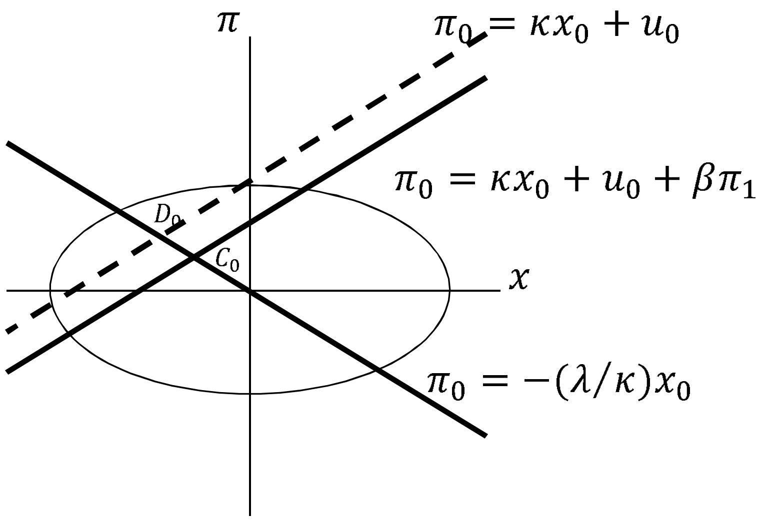
VAN ZANDWEGHE, W., AND A. L. WOLMAN (2010): "Discretionary Monetary Policy in the Calvo Model," Working Paper RWP 10-06 Federal Reserve Bank of Kansas City.

WOODFORD, M. (1999): "Commentary: How Should Monetary Policy Be Conducted in an Era of Price Stability?," *Federal Reserve Bank of Kansas City, New Challenges for Monetary Policy.*

WOODFORD, M. (2003): *Interest and Prices: Foundations of a Theory of Monetary Policy.* Princeton University Press, Princeton, NJ.

Figure 1: Stabilization bias and time inconsistency.

a. Period 0



b. Period 1

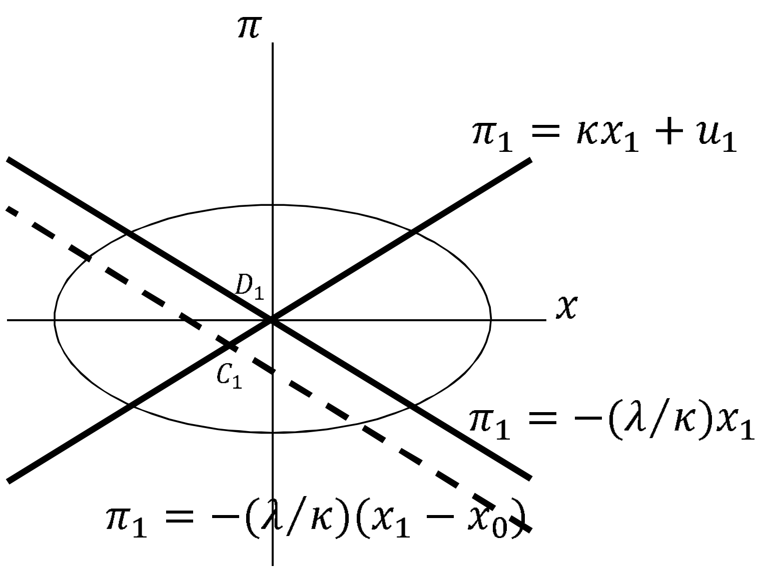


Figure 2: Impulse response functions: $\beta = 0.9$.

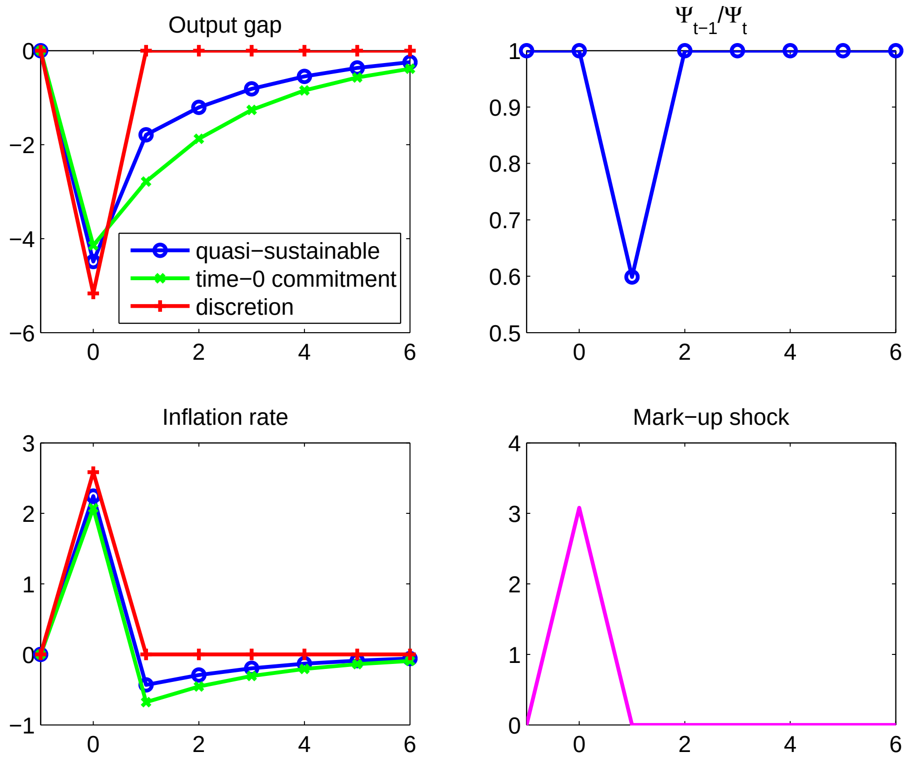
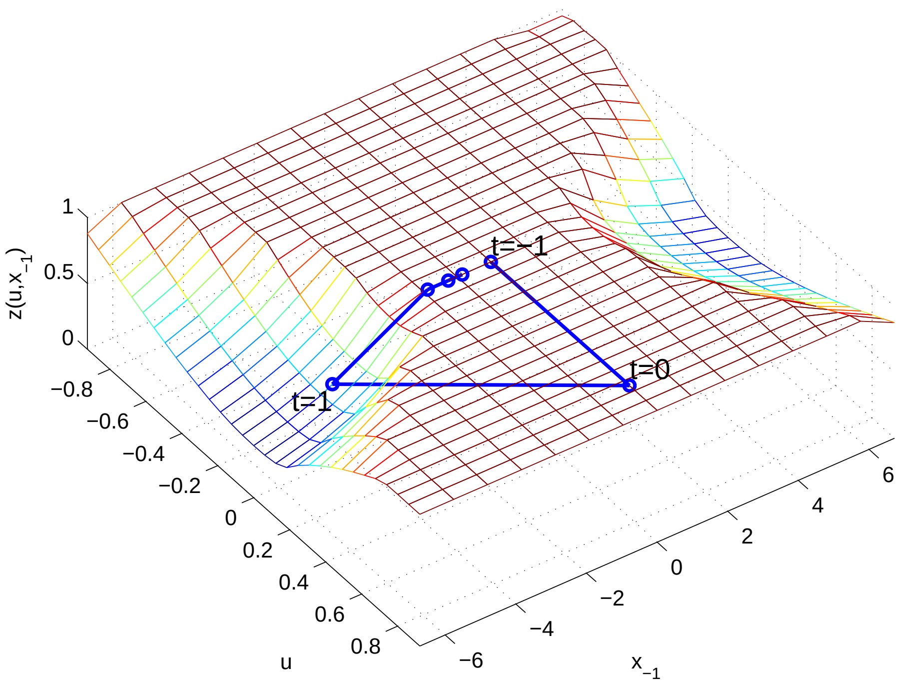
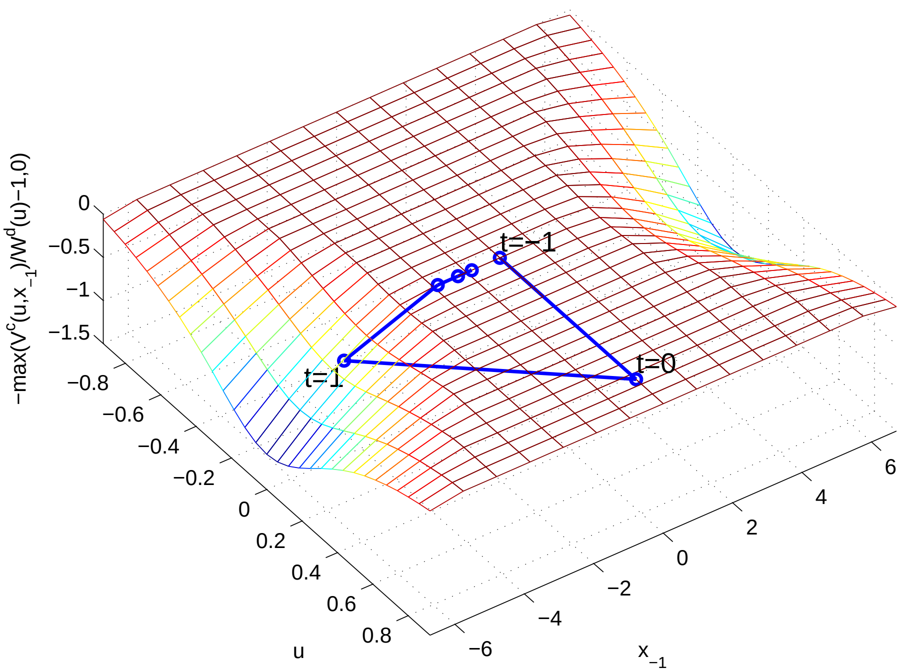


Figure 3: Policy function of $z(u, x_{-1})$ and difference in social welfare: $\beta = 0.9$.

a. $z(u, x_{-1})$



b. difference in social welfare $(-\max\{V(u, x_{-1})/W^d(u) - 1, 0\})$



Notes: The state space consists of the markup shock U and the lagged output gap X. The solid line with the circle indicates responses of the optimal quasi-sustainable policy with the initial values $(x_{-1}, u_0) = (0, 5\sigma)$.

Figure 4: Policy functions of $\pi(u,0)$ and $x(u,0)$: $\beta = 0.9$.

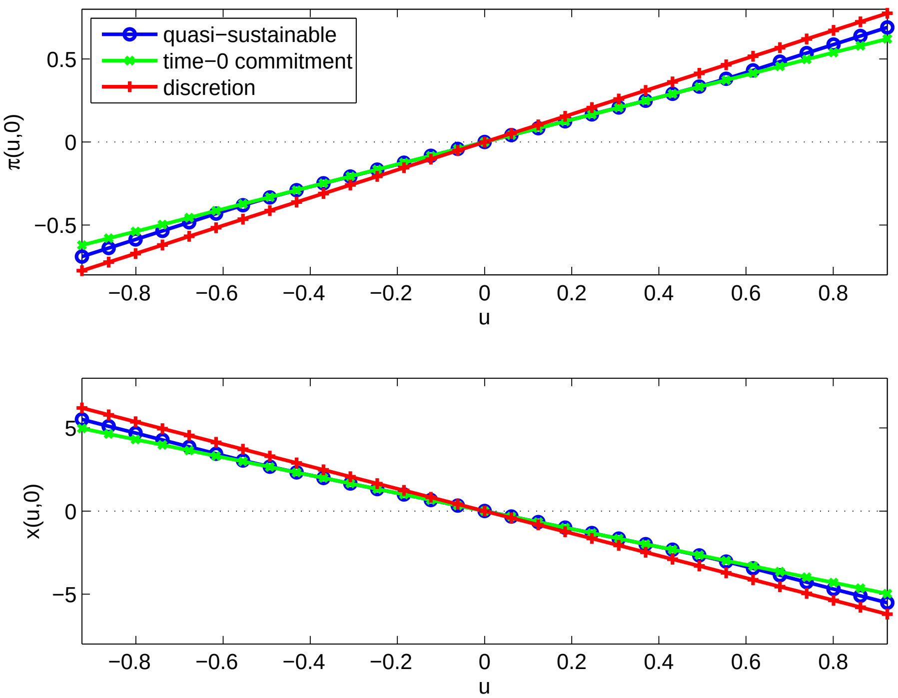


Figure 5: Sample simulated path: $\beta = 0.9$.

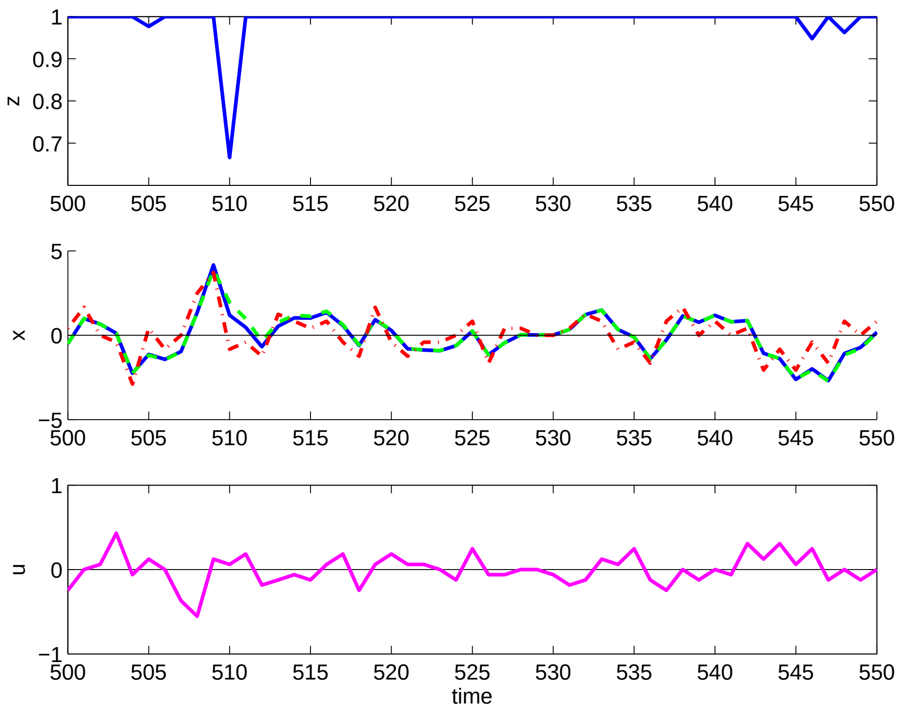


Table 1: Parameter values.

Parameters		Values	Sources
β	Discount factor	0.9 or 0.9913	3.5% annual real rate (for $\beta = 0.9913$)
κ	Slope of the Phillips curve	0.024	Woodford (2003)
λ	Weight to the output gap	$0.048/4^2$	Woodford (2003)
ρ	AR(1) parameter of markup shock	0.0	Adam and Billi (2006)
σ_ε	Std. deviation of markup shock (%, q/q)	0.154	Adam and Billi (2006)

Table 2: Simulation results with baseline parameters.

	$\beta = 0.9$				
	σ_π	σ_x	$\bar{V}$	$V(0,0)$	$P_{\{z<1\}}$
Sustainable	0.4556	1.1228	-0.1675	-0.1456	6.79
Commitment	0.4561	1.1289	-0.1682	-0.1454	—
Discretion	0.5203	1.0405	-0.2017	-0.1815	—
	$\beta = 0.9913$				
	σ_π	σ_x	$\bar{V}$	$V(0,0)$	$P_{\{z<1\}}$
Sustainable	0.4435	1.0612	-1.8013	-1.7778	0.00
Commitment	0.4435	1.0612	-1.8013	-1.7778	—
Discretion	0.5203	1.0405	-2.3179	-2.2972	—

Notes: 100 simulations are perfomed to calculate the mean of standard deviations in simulations for each policy. The length of periods in each simulation is 1,000, and 50 initial periods are discarded to eliminate the effect of initial values. The same pseudo-random numbers are used in simulations for each policy.

Table 3: Results with alternative parameters.

a. Probability of binding constraints, $P_{\{z<1\}}$ (%)

	κ	λ		
		0.001	0.01	0.0625
$\rho = 0$	0.005	0.00	0.53	1.35
	0.01	0.00	0.01	0.36
	0.05	0.00	0.00	0.00
	κ	λ		
		0.001	0.01	0.0625
$\rho = 0.35$	0.005	0.00	0.57	5.45
	0.01	0.00	0.00	0.33
	0.05	0.00	0.00	0.00

b. Ratio of unconditional means of the output gap, σ_x^c/σ_x^d

	κ	λ		
		0.001	0.01	0.0625
$\rho = 0$	0.005	1.71	3.25	5.61
	0.01	1.19	2.20	3.70
	0.05	0.85	0.97	1.50
	κ	λ		
		0.001	0.01	0.0625
$\rho = 0.35$	0.005	2.20	4.51	7.97
	0.01	1.41	2.94	5.17
	0.05	0.90	1.10	1.89

c. Lower bounds of discount factor (with unconditional means), $\underline{\beta}$

	κ	λ		
		0.001	0.01	0.0625
$\rho = 0$	0.005	0.838	0.944	0.977
	0.01	0.719	0.893	0.955
	0.05	0.493	0.626	0.802
	κ	λ		
		0.001	0.01	0.0625
$\rho = 0.35$	0.005	0.844	0.945	0.977
	0.01	0.734	0.895	0.956
	0.05	0.511	0.649	0.810

Table 4: Simulation results with alternative parameters.

(i) $\rho = 0.0$

	κ	σ_π			σ_x			$\bar{V}$			$V(0,0)$		
		λ			λ			λ			λ		
		0.001	0.01	0.0625	0.001	0.01	0.0625	0.001	0.01	0.0625	0.001	0.01	0.0625
Sustainable	0.005	0.5521	0.5996	0.6134	1.2908	0.2442	0.0571	-2.0717	-2.3064	-2.3720	-2.0408	-2.2729	-2.3437
	0.01	0.4883	0.5772	0.6041	1.6713	0.3363	0.0873	-1.7695	-2.1956	-2.3285	-1.7428	-2.1623	-2.2958
	0.05	0.1849	0.4225	0.5345	1.8760	0.6007	0.1793	-0.5657	-1.4767	-1.9867	-0.5578	-1.4549	-1.9563
Commitment	0.005	0.5521	0.5996	0.6132	1.2908	0.2467	0.0642	-2.0717	-2.3078	-2.3762	-2.0398	-2.2727	-2.3413
	0.01	0.4883	0.5772	0.6041	1.6713	0.3363	0.0891	-1.7695	-2.1957	-2.3308	-1.7428	-2.1619	-2.2955
	0.05	0.1849	0.4225	0.5345	1.8760	0.6007	0.1793	-0.5657	-1.4767	-1.9867	-0.5578	-1.4549	-1.9563
Discretionary	0.005	0.6050	0.6186	0.6199	0.7563	0.0773	0.0124	-2.3452	-2.3978	-2.4028	-2.3212	-2.3733	-2.3782
	0.01	0.5638	0.6140	0.6192	1.4095	0.1535	0.0248	-2.1853	-2.3800	-2.3999	-2.1629	-2.3556	-2.3754
	0.05	0.1772	0.4961	0.5963	2.2149	0.6202	0.1193	-0.6868	-1.9230	-2.3113	-0.6798	-1.9034	-2.2877

(ii) $\rho = 0.35$

	κ	σ_π			σ_x			$\bar{V}$			$V(0,0)$		
		λ			λ			λ			λ		
		0.001	0.01	0.0625	0.001	0.01	0.0625	0.001	0.01	0.0625	0.001	0.01	0.0625
Sustainable	0.005	0.8110	0.9488	0.9929	2.6696	0.5503	0.1310	-4.8230	-5.9287	-6.2686	-4.7109	-5.7939	-6.1621
	0.01	0.6555	0.8812	0.9631	3.1086	0.7269	0.1981	-3.6521	-5.3821	-6.0421	-3.5709	-5.2551	-5.9105
	0.05	0.1815	0.5225	0.7651	2.3709	1.0118	0.3598	-0.7681	-2.7298	-4.4679	-0.7540	-2.6723	-4.3647
Commitment	0.005	0.8110	0.9488	0.9927	2.6696	0.5547	0.1475	-4.8230	-5.9347	-6.2952	-4.7101	-5.7934	-6.1503
	0.01	0.6555	0.8812	0.9632	3.1086	0.7270	0.2018	-3.6521	-5.3822	-6.0534	-3.5709	-5.2543	-5.9098
	0.05	0.1815	0.5225	0.7651	2.3709	1.0118	0.3598	-0.7681	-2.7298	-4.4679	-0.7540	-2.6723	-4.3647
Discretionary	0.005	0.9782	1.0117	1.0150	1.2227	0.1265	0.0203	-6.1298	-6.4136	-6.4413	-6.0394	-6.3190	-6.3463
	0.01	0.8808	1.0003	1.0131	2.2021	0.2501	0.0405	-5.3339	-6.3163	-6.4254	-5.2553	-6.2232	-6.3307
	0.05	0.2105	0.7346	0.9570	2.6308	0.9182	0.1914	-0.9690	-4.2158	-5.9534	-0.9547	-4.1536	-5.8656