Online Appendix to: "Optimal Monetary Policy with Labor Market Frictions: The Role of the Wage Channel"

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Proof of Proposition 3

This section is largely based on Thomas (2007, working paper version). The common part of the equilibrium conditions between EB and RTM is used so that the correct LQ approximation has the same form between EB and RTM.

Household utility The period-by-period household utility $U_t = \frac{c_t^{1-\sigma}}{1-\sigma} - \kappa n_t \frac{h_t^{1+\phi}}{1+\phi}$ is approximated as up to second order:

$$\begin{aligned}
U_t - U &= c^{-\sigma} c \left(\frac{c_t - c}{c}\right) + (-\sigma) c^{-\sigma-1} c^2 \frac{1}{2} \left(\frac{c_t - c}{c}\right)^2 \\
&- \kappa n h^{\phi} h \left(\frac{h_t - h}{h}\right) - \phi \kappa n h^{\phi-1} h^2 \frac{1}{2} \left(\frac{h_t - h}{h}\right)^2 \\
&- \kappa \frac{h^{1+\phi}}{1+\phi} n \left(\frac{n_t - n}{n}\right) - \kappa h^{\phi} h n \left(\frac{h_t - h}{h}\right)\left(\frac{n_t - n}{n}\right) + O^3, \\
&= c^{1-\sigma}\left(\hat{c}_t + \frac{1-\sigma}{2}\hat{c}_t^2\right) \\
&- \kappa n h^{1+\phi}\left(\hat{h}_t + \frac{1}{1+\phi}\hat{n}_t + \frac{1+\phi}{2}\hat{h}_t^2 + \frac{1}{2(1+\phi)}\hat{n}_t^2 + \hat{n}_t\hat{h}_t\right) + O^3, \qquad (1)
\end{aligned}$$

where O^3 includes terms in third order or more. Note that we used $\frac{x_t - x}{x} \approx \hat{x}_t + \frac{1}{2}\hat{x}_t^2$ where $\hat{x}_t = \log(x_t/x)$.

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Resource constraint The resource constraint is approximated as

$$\begin{aligned}\left(\hat{c}_t + \frac{1-\sigma}{2}\hat{c}_t^2\right) &= -\frac{\psi}{2c}\pi_t^2 - \frac{\sigma}{2}\hat{c}_t^2 + \frac{y}{2c}\hat{y}_t^2 + \frac{y}{c}\left(\hat{a}_t + \alpha\hat{h}_t + \hat{n}_t\right) \\ &\quad -\frac{nb}{c}\left(\hat{n}_t + \frac{1}{2}\hat{n}_t^2\right) - \frac{c_v v}{c}\left(\hat{v}_t + \frac{1}{2}\hat{v}_t^2\right) + \text{t.i.p.} + O^3.\end{aligned}$$

where t.i.p. stands for terms irrelevant to policies. When the steady state is efficient, $\alpha y c^{-\sigma} = \kappa n h^{1+\phi}$ holds. By substituting them into equation (1), we have

$$\begin{aligned}U_t - U &= -c^{1-\sigma}\left[\frac{\psi}{2c}\pi_t^2 + \frac{\sigma}{2}\hat{c}_t^2 - \frac{y}{2c}\hat{y}_t^2 + \frac{nb}{2c}\hat{n}_t^2 + \frac{\alpha}{1+\phi}\frac{y}{c}\frac{1}{2}\left((1+\phi)\hat{h}_t + \hat{n}_t\right)^2\right] \\ &\quad + nc^{-\sigma}\left[\left(\frac{y}{n} - \frac{\kappa h^{1+\phi}c^{\sigma}}{1+\phi} - b\right)\hat{n}_t - c_v\frac{v}{n}\left(\hat{v}_t + \frac{1}{2}\hat{v}_t^2\right)\right] + \text{t.i.p.} + O^3. \qquad (2)\end{aligned}$$

The second term of equation (2) includes linear terms, which is got rid of by using second order approximation of the equilibrium conditions as follows.

The Beveridge curve $n_t = (1-\rho)n_{t-1} + \bar{m}u_t^{\xi}v_t^{1-\xi}$ is approximated as (note that $\rho n = m$)

$$\begin{aligned}\left(\hat{n}_t + \frac{1}{2}\hat{n}_t^2\right) &= (1-\rho)\left(\hat{n}_{t-1} + \frac{1}{2}\hat{n}_{t-1}^2\right) \\ &\quad + \rho\left(\xi\hat{u}_t + (1-\xi)\hat{v}_t + \frac{1}{2}\left[\xi\hat{u}_t + (1-\xi)\hat{v}_t\right]^2\right) + O^3.\end{aligned}$$

Also, $u_t = 1 - (1-\rho)n_{t-1}$ is approximated as (note that $p = \rho n/u$)

$$\hat{u}_t = -\frac{1}{2}\hat{u}_t^2 - \frac{p(1-\rho)}{\rho}\left(\hat{n}_{t-1} + \frac{1}{2}\hat{n}_{t-1}^2\right) + O^3.$$

By rearranging these equations, we have

$$\begin{aligned}-\beta^{-1}\left(\hat{n}_{t-1} + \frac{1}{2}\hat{n}_{t-1}^2\right) &= \left[-\beta^{-1} + (1-\rho)(1-p\xi)\right]\left(\hat{n}_{t-1} + \frac{1}{2}\hat{n}_{t-1}^2\right) \\ &\quad - \frac{\rho\xi}{2}\hat{u}_t^2 + \rho(1-\xi)\hat{v}_t + \frac{\rho}{2}\left[\xi\hat{u}_t + (1-\xi)\hat{v}_t\right]^2 + O^3.\end{aligned}$$

Solving it forward,

$$\begin{aligned}(1-\rho)(1-p\xi)\left(\hat{n}_{-1} + \frac{1}{2}\hat{n}_{-1}^2\right) &= E_0\sum_{t=0}^{\infty}\left\{\frac{\rho\xi}{2}\hat{u}_t^2 - \rho(1-\xi)\hat{v}_t - \frac{\rho}{2}\left[\xi\hat{u}_t + (1-\xi)\hat{v}_t\right]^2\right. \\ &\quad \left. + \left[1 - \beta(1-\rho)(1-p\xi)\right]\left(\hat{n}_t + \frac{1}{2}\hat{n}_t^2\right)\right\} + O^3.\end{aligned}$$

Assuming that the initial values are in the steady state, $\hat{n}_{-1} = 0$,

$$
\begin{aligned}
& E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left[1 - \beta(1-\rho)(1-\xi p)\right] \hat{n}_t - \rho(1-\xi)\hat{v}_t \right. \\
= \quad & E_0 \sum_{t=0}^{\infty} \beta^t \left\{ - \left[1 - \beta(1-\rho)(1-\xi p)\right] \frac{1}{2}\hat{n}_t^2 - \frac{\rho\xi}{2}\hat{u}_t^2 + \frac{\rho}{2}\left[\xi \hat{u}_t + (1-\xi)\hat{v}_t\right]^2 \right\} \\
& + \text{t.i.p.} + O^3. \qquad (3)
\end{aligned}
$$

When the steady state is efficient, $[1-\beta(1-\rho)(1-\xi p)] = \frac{(1-\xi)q}{c_v}\left(\frac{y}{n} - \frac{\kappa h^{1+\phi} c^{\sigma}}{1+\phi} - b\right)$ holds. Then, the linear terms in equation (3) is propotional to those in equation (2) (note that $\rho/q = v/n$),

$$
\begin{aligned}
& E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left[1 - \beta(1-\rho)(1-\xi p)\right] \hat{n}_t - \rho(1-\xi)\hat{v}_t \right\}, \\
= \quad & E_0 \sum_{t=0}^{\infty} \beta^t \frac{(1-\xi)q}{c_v} \left\{ \left(\frac{y}{n} - \frac{\alpha y/n}{1+\phi} - b \right) \hat{n}_t - \frac{c_v v}{n}\hat{v}_t \right\}.
\end{aligned}
$$

Then, the second term of equation (2) can be written by quadratic terms only:

$$
\begin{aligned}
& E_0 \sum_{t=0}^{\infty} \beta^t n c^{-\sigma} \left\{ \left(\frac{y}{n} - \frac{\alpha y/n}{1+\phi} - b \right) \hat{n}_t - \frac{c_v v}{n}\left(\hat{v}_t + \frac{1}{2}\hat{v}_t^2 \right) \right\}, \\
= \quad & E_0 \sum_{t=0}^{\infty} \beta^t n c^{-\sigma} \frac{c_v}{(1-\xi)q} \left\{ - \left[1 - \beta(1-\rho)(1-\xi p)\right] \frac{1}{2}\hat{n}_t^2 - \frac{\rho\xi}{2}\hat{u}_t^2 \right. \\
& \left. + \frac{\rho}{2}\left[\xi\hat{u}_t + (1-\xi)\hat{v}_t\right]^2 - (1-\xi)\frac{qv}{n}\frac{1}{2}\hat{v}_t^2 \right\} + \text{ t.i.p.} + O^3.
\end{aligned}
$$

Finally, by substituting it into equation (2) and arranging the terms, we have

$$
\begin{aligned}
V_0 - V \quad = \quad & E_0 \sum_{t=0}^{\infty} \beta^t (U_t - U), \\
& = -\frac{c^{1-\sigma}}{2} E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{\psi}{c}\pi_t^2 + \sigma \hat{c}_t^2 - \frac{y}{c}(\hat{y}_t^2 - \hat{n}_t^2) \right. \\
& - \frac{\alpha y/c}{1+\phi}\left(\hat{n}_t^2 - \left[(1+\phi)\hat{h}_t + \hat{n}_t\right]^2 \right) \\
& \left. + \frac{c_v v/c}{(1-\xi)}\left(\xi\hat{u}_t^2 + (1-\xi)\hat{v}_t^2 - \left[\xi\hat{u}_t + (1-\xi)\hat{v}_t\right]^2 \right) \right] + \text{t.i.p.} + O^3.
\end{aligned}
$$